Module 4 – Independence and Bayes' Theorem

Reading: Sections 2.6 – 2.8

Practical Approach to the Multiplication Rule
Going back to the previous example, suppose you select a tool at random and repeat this process five times (without replacement). What is the probability that all five tools you selected are Black & Decker?

- Remember, there are 50 tools in total, 10 are Craftsman and 40 are B&D
- Here, students often get confused and write $P = 5 * \left(\frac{40}{50}\right)$ or $P = \left(\frac{40}{50}\right)^5$
- Neither of these are correct. Instead, let's focus on a practical approach to the multiplication rule.
- Our approach will consist of looking at each step individually and then multiplying the probabilities at each step.

Step 1: Select a Black & Decker

$$P_1 = \left(\frac{40}{50}\right)$$

Step 2: Select another Black & Decker. At this point, we no longer have 50 tool sin the set, we have 49 tools. Also, we no longer have 40 B&D tools in the set, we have 39, so:

$$P_2 = \left(\frac{39}{49}\right)$$

Step 3: Select another Black & Decker.

$$P_3 = \left(\frac{38}{48}\right)$$

Step 4: Select another Black & Decker.

$$P_4 = \left(\frac{37}{47}\right)$$

Step 5: Select another Black & Decker.

$$P_5 = \left(\frac{36}{46}\right)$$

Apply the multiplication rule:

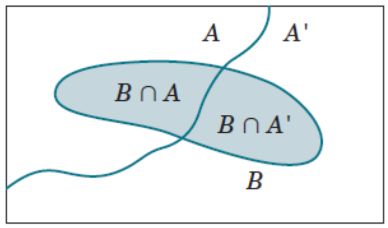
$$P = \left(\frac{40}{50}\right)\left(\frac{39}{49}\right)\left(\frac{38}{48}\right)\left(\frac{37}{47}\right)\left(\frac{36}{46}\right) \approx 0.311$$

We will follow this approach in class for most multiplication problems that involve sampling without replacement.

Total Probability Rule
For any two events A and B,

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$$

Graphically, this can be visualized with the following sample space:



Independence

We say that the events A and B are independent if the outcome of one event does not influence the probability of the other event. Statistically, two events are independent if:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \cap B) = P(A)P(B)$

Note that independent events are not mutually exclusive, because for mutually exclusive events $P(A \cap B) = \emptyset$.

It's important to understand when two events are or aren't independent in order to develop conclusions and interpretations of data.

Discussion questions:
- Is a person's financial status independent of their propensity to commit a crime?
- Is a person's geographical location independent of their native language?
- Is a car's color independent of its engine performance?

Example: Sampling with Replacement
The most common example of independence comes from sampling with replacement. Let's repeat the example of the tools in a bin, but this time, suppose that before selecting a new tool, you replace the original tool in the bin. What is the probability that all five tools you select are Black & Decker?

- Once again, we divide our experiment into steps.
- Step 1: select a B&D tool: $P_1 = \frac{40}{50}$
- Step 2: replace the B&D tool and select another B&D tool. Here, because the tool was replaced, we now have 40 B&D tools and 50 tools in total, so $P_2 = \frac{40}{50}$
- Steps 3 – 5: replace the tool and select another B&D tool. $P_3 = P_4 = P_5 = \frac{40}{50}$
- Apply the multiplication rule.

$$P = \left(\frac{40}{50}\right)\left(\frac{40}{50}\right)\left(\frac{40}{50}\right)\left(\frac{40}{50}\right)\left(\frac{40}{50}\right) \approx 0.328$$

Bayes' Theorem

- Thomas Bayes was a Presbyterian minister
- Credited with the development of Bayesian probability, which interprets probability as a reasonable expectation (instead of a frequency) of an event, based on prior knowledge.
- Bayes' methods update prior knowledge (initial probability) into new knowledge (posterior probability) based on new available data (evidence)

We can mathematically derive Bayes' theorem based on the multiplication rule:

$$P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A)$$

to obtain:

$$\boldsymbol{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Also, recall the total probability rule: $P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$ which can be inserted into Baye's theorem in the case where multiple events are present.

Example: Medical Diagnosis

A test is being administered to diagnose an illness. The probability that the test correctly identifies someone with the illness as positive (sensitivity) is 0.99. The probability that the test correctly identifies someone without the illness as negative (specificity) is 0.95. The incidence of the illness in the general population is 0.0001.

If you take the test and the result is positive, what is the probability that you have the illness?

- First, let's express these data in statistical terms. Let D be the event that you have the illness, S be the event that your test signals positive.
- This means we are looking for $P(D|S)$

- Bayes' theorem states that

$$P(D|S) = \frac{P(S|D)P(D)}{P(S)}$$

- The total probability rule states that

$$P(S) = P(S \cap D) + P(S \cap D') = P(S|D)P(D) + P(S|D')P(D')$$

- From the given data, we know that:
  - $P(S|D) = 0.99$, the probability that you test positive given that you have the illness
  - $P(D) = 0.0001$, the probability that you have the illness (incidence of illness in the general population)
  - $P(S|D') = 1 - 0.95 = 0.05$, the probability that you test positive if you don't have the illness (this is found by taking one minus the probability that you test negative if you don't have the illness).

- $P(D') = 1 - 0.0001 = 0.9999$, the probability that you don't have the illness (one minus the incidence of illness in the general population).

- Solving:

$$P(D|S) = \frac{P(S|D)P(D)}{P(S|D)P(D) + P(S|D')P(D')} = \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.05)(0.9999)} = \frac{1}{506} = 0.002$$

Practical Approach to Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

We can approach Bayes' theorem from a practical perspective through Bayesian interpretation. The equation above tells us that we will start with a prior probability $P(A)$ and then update it with new evidence $\frac{P(B|A)}{P(B)}$ to obtain a posterior probability $P(A|B)$.

Here $\frac{P(B|A)}{P(B)}$ is considered new evidence because it looks at the support B provides for A.

Homework: Problems 2.6.5, 2.6.9, 2.7.5, 2.7.11, 2.8.1

Project Part 1: Use a statistical analysis to determine if a person's economic status is independent of their propensity to commit a crime in the United States.

Note: This is not about personal opinions. You are expected to look at crime/demographics data, set up a sample space and events, and then use the equations of Independence to prove or disprove the point.

Project Part 2: Use statistical analysis to determine the probability that you have COVID-19 if you test positive in the United States.

Note: This is not about personal opinions. You are expected to look at medical/population data, set up a sample space and events, and then use Bayes' Theorem to calculate your probability.

References

Bayes, *An Essay towards solving a Problem in the Doctrine of Chances*
Montgomery, Runger, *Applied Statistics and Probability for Engineers*
McGrayne, *The Theory that Would not Die*