Module 3 – Introduction to Probability

Reading: Sections 2.3 – 2.6

What is Probability?

- Try to define it in your own words. Try to define it with an example.
- probability is the likelihood of an outcome
- What is likelihood?

Subjective Probability

- degree of belief
- "I am 99% sure I turned off the oven"
- This is not what we study in this course

Objective Probability

- Based on repeated replications of an experiment
- The limiting value of the proportion of times an outcome occurs in an experiment.

Equally Likely Outcomes

- Let's start with a simple concept:
- Suppose that an experiment has N outcomes where each outcome has the same probability.
- In other words, each outcome is equally likely to occur.
- Examples: flipping a coin, rolling a die, picking a card at random
- The probability of each outcome is $\frac{1}{N}$.

Example: Flipping a Coin

What is the probability that a coin lands on heads?

- There are two outcomes: $S = \{H, T\}$
- Each outcome is equally likely to occur
- The probability is $\frac{\#\ of\ outcomes\ with\ H}{total\ \#\ of\ outcomes}\ \frac{1}{2} = 0.5$ or 50%

What is the probability that a coin will land on heads at least once if you flip a coin twice?

- The sample space is now: $S = \{HH, HT, TH, TT\}$
- Each outcome is equally likely to occur
- The probability is $\frac{\#\ of\ outcomes\ with\ H}{total\ \#\ of\ outcomes}\ \frac{3}{4} = 0.75$ or 75%

Example: Full House

In poker, a full house is defined as having three cards of one rank and two cards of another (example: three 7's and two 5's). What is the probability of getting a full house if cards are drawn randomly?

- Since we are drawing cards at random, we can assume that each card is equally likely to be selected. Therefore, our probability will be:

$$\frac{\#\ of\ outcomes\ in\ F}{\#\ of\ outcomes\ in\ S}$$

where S is the sample space and F is the event in the sample space that includes all full house arrangements. In the previous module, we found that the number of possible arrangements in poker is $\binom{52}{5}$.

- For the numerator, we have 13 possibilities of a card (A, 2, 3, …, 10, J, Q, K). For each rank, there are 4 cards (hearts, cloves, swords, diamonds), but we only need to choose 3:

$$13 * \binom{4}{3}$$

- Then, we have 12 possible cards to choose from (A to K minus the card already chosen). For each rank, there are 4 cards but we only need to choose 2:

$$12 * \binom{4}{2}$$

- Then, we apply the multiplication rule:

$$13 * \binom{4}{3} * 12 * \binom{4}{2}$$

- Therefore, the probability of landing a full house is:

$$P(F) = \frac{13 * \binom{4}{3} * 12 * \binom{4}{2}}{\binom{52}{5}} = \frac{3{,}744}{2{,}598{,}960}$$

Unequal Probability

- Not all experiments have equally likely outcomes.
- Example: What is the probability that there is life in Neptune?
- The sample space is $S = \{L, N\}$ (L=life, N=no life)
- If we assume equally likely outcomes, we would get a probability of 50%.
- That does not seem right.

Probability of Events

- Recall that the sample space is a set and an event is a subset of the sample space.
- Let E be an event within sample space S
- The probability of E, or $P(E)$ is the sum of the probabilities of all outcomes in E.

Example: Probability of an Event

Suppose your experiment has four possible outcomes with the following probabilities:

Outcome	Probability
a	0.1
b	0.3
c	0.5
d	0.1

Let's define three events: $A = \{a, b\}$, $B = \{b, c, d\}$, $C = \{d\}$. What is the probability of each event?

- $P(A) = P(a) + P(b) = 0.1 + 0.3 = 0.4$
- $P(B) = P(b) + P(c) + P(d) = 0.9$
- $P(C) = P(d) = 0.1$

Axioms of Probability
We define probability in this course through the following axioms:

- The probability of a sample space is 1. $P(S) = 1$
- The probability of any event is a number between 0 and 1. $0 \leq P(E) \leq 1$
- If two events are mutually exclusive, then the probability of the union of the events is equal to the sum of the probabilities of each event. If $E_1 \cap E_2 = \emptyset$, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

From these axioms, we can also derive the following:

- $P(\emptyset) = 0$
- $P(E') = 1 - P(E)$

Finally, if event E_1 is a subset of event E_2, then:

- $P(E_1) \leq P(E_2)$

Probability of a Union

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
- Four or more events follows the same pattern, but we will not see them in class.

Conditional Probability
- Sometimes, probabilities are reevaluated as additional information becomes available
- This is known as <u>conditional probability</u>
- The conditional probability of event B occurring given that event A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- For short, we say "probability of B given A"

Example: Defective Parts and Surface Flaws
Suppose you are inspecting parts for a manufacturer. Your inspection consists on determining if the parts have surface flaws (visible defects, regardless of whether or not the part works) and if they are functionally defective (whether or not they work). Your investigation of 400 parts results in the following table:

		Surface Flaws		
		Yes	**No**	**Total**
Defective	**Yes**	10	18	28
	No	30	342	372
	Total	40	360	400

What is the probability that if you pick a part at random, that part is defective? What is the probability that a part is defective if it has surface flaws? What is the probability that a part is defective even though it doesn't have surface flaws?

Set up the problem:

- First, let's define our sample space: S includes all 400 parts, and their outcome (if they have or don't have surface flaws and if they are or aren't defective)
- We can create events of our sample space. Let F be the event that includes the parts that have surface flaws. In this case, F includes 40 parts. Also, F' (the complement of F) would consists of all parts that don't have surface flaws. In this case, F' has 360 parts.
- Let D be the event that includes the parts that are defective. In this case, D would have 28 parts and D' would have 372 parts.
- Since we are selecting parts at random to assess probability, then we can assume that each part is equally likely to be picked.

Solve the problem:

- The probability that a part is defective is equal to the number of defective parts over the total number of parts:
$$P(D) = \frac{28}{400} = 0.07$$
- The conditional probability that a part is defective given that it has surface flaws is:
$$P(D \mid F) = \frac{P(F \cap D)}{P(F)} = \frac{10}{40} = 0.25$$
- The conditional probability that a part is defective given that it doesn't have surface flaws is:
$$P(D \mid F') = \frac{P(F' \cap D)}{P(F')} = \frac{18}{360} = 0.05$$

Multiplication Rule
The probability of an intersection of two events is a function of the conditional probability involving said events. This can be found by rearranging the conditional probability equation:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Longrightarrow P(A \cap B) = P(B \mid A)P(A)$$

Also, remember that $P(A \cap B) = P(B \cap A)$, therefore:

$$P(B \mid A)P(A) = P(A|B)P(B)$$

This is known as the multiplication rule for probabilities.

Random Samples (Example: Tools in a Bin)
Suppose you have 50 tools in a bin, of which 10 are Craftsman and 40 are Black & Decker. If you pick one tool at random, what is the probability that that tool is a Craftsman?

- For simplicity, let C be the event that includes all Craftsman tools and B be the event that includes all Black & Decker tools.
- Since we are selecting a tool at random, then we assume that all tools are equally likely to be selected.
- The probability that the tool is a Craftsman is:

$$P(C) = \frac{10}{50} = 0.2$$

Now, suppose that you select a second tool at random (without replacement). What is the probability that the second tool is a Black & Decker?

- Here we are sampling without replacement, which means that the total number of tools is now 49.
- Statistically, what we are looking for is the conditional probability that you select a Black & Decker given that you already selected a Craftsman:

$$P(B|C) = \frac{P(C \cap B)}{P(C)}$$

- But we don't know $P(C \cap B)$. It's okay, we can still manually find this probability.
- Since each tool is equally likely to be selected, then the probability of selecting a B&D is equal to the number of B&D tools divided by the total number of tools:

$$P(B|C) = \frac{40}{49} \approx 0.816$$

Therefore, what is the probability that the first tool you selected is a Craftsman and the second tool you selected is a Black & Decker?

- Here we are being asked for $P(C \cap B)$, which we can find using the multiplication rule:

$$P(C \cap B) = P(B|C)P(C) = \left(\frac{40}{49}\right)\left(\frac{10}{50}\right) = \frac{8}{49} \approx 0.163$$

Finally, note that since $P(C \cap B) = P(B \cap C)$ then we should expect that the probability of first selecting a Craftsman and then a B&D is the same as the probability of first selecting a B&D and then a Craftsman:

$$P(B \cap C) = P(C|B)P(B) = \left(\frac{10}{49}\right)\left(\frac{40}{50}\right) = \frac{8}{49} \approx 0.163$$

Probability and Quantum Physics

Maxwell-Boltzmann statistics describe the distribution of particles over various energy states. In the 1920's, Satyendra Nath Bose proved that this description was not applicable to quantum states, because photons are indistinguishable from each other. He sent his paper to Albert Einstein, who agreed with him and further worked to develop Bose-Einstein statistics.

We have talked about flipping a coin twice. In Maxwell-Boltzmann statistics, the particles (coins) are distinguishable, and so if we want to determine their distribution (heads/tails) across two energy states (flipping the coins), then there are 4 outcomes: $S = \{HH, HT, TH, TT\}$.

However, in the quantum state, the particles (coins) are indistinguishable. This means that in order to determine their distribution (heads/tails), order doesn't matter! Across two energy states (flips), there are 3 outcomes: $S = \{H^2T^0, H^1T^1, H^0T^2\}$. Where $HT = TH$.

In summary, coins, which are large enough so that quantum effects are negligible, follow Maxwell-Boltzmann statistics. Therefore, the probability of flipping two coins and landing on heads twice is:

$$P(HH) = \frac{1}{4} = 0.25$$

But, in an alternate universe where coins behave like photons, the probability of flipping two coins and landing on heads twice is:

$$P(HH) = \frac{1}{3} = 0.33$$

This is the world of quantum physics, where things don't behave the way we're used to seeing with classical mechanics. As you can see, statistics are a crucial part of understanding quantum physics.

Homework: Problems 2.3.3, 2.3.7, 2.3.11, 2.3.12, 2.3.15, 2.4.7, 2.5.1, 2.5.9

References

Blitzstein, Hwang, *Introduction to Probability*
Montgomery, Runger, *Applied Statistics and Probability for Engineers*