Module 2 – Sets and Counting

Reading: Sections 2.1 – 2.2

Example: Electronics
Suppose we are measuring the resistance in a thin copper wire. The variables we control are:

- voltage
- current
- geometry of the wire (length, diameter)

Even while keeping these variables constant, our output has variability. Why?

Are there any other variables we did not consider?

- ambient temperature
- gage variations
- impurities in the chemical composition of the wire
- current source drifts

These variables that we cannot control (and sometimes cannot measure) are called noise.

The outcome of an experiment is a function of the inputs, which include control variables and noise.

Sample Spaces

- discrete: finite or countably infinite number of outcomes
  - flipping a coin, rolling a die
- continuous: infinite outcomes in an interval
  - mass, velocity

Example: Flip a Coin Twice
If H and T represent heads and tails, respectively, what is the sample space of this experiment?

$$S = \{HH, HT, TH, TT\}$$

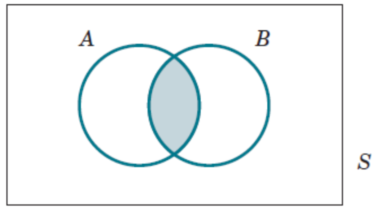
Math Review: Set Operators
Let A and B be two events in sample space S.

- Union, $A \cup B$: all outcomes in either A or B
- Intersection, $A \cap B$: all outcomes in both A and B
- Complement, A': all outcomes that are not in A.

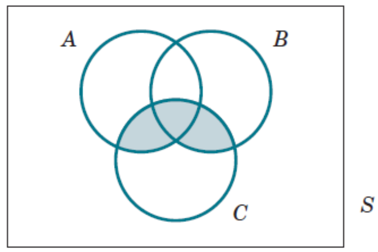
Graphically, we can use Venn Diagrams to represent continuous sample spaces.

Examples: Venn Diagrams

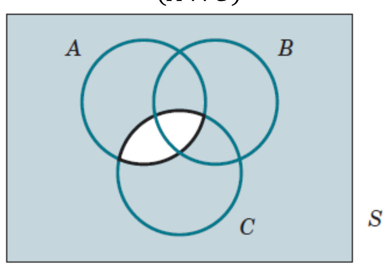
$A \cap B$



$(A \cup B) \cap C$



$(A \cap C)'$



Math Review: Laws for Set Operations

- mutually exclusive events: don't share any outcomes, $A \cap B = \emptyset$ (empty set)
- $(A')' = A$
- Distributive law:
    - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
    - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- DeMorgan's law
    - $(A \cup B)' = A' \cap B'$
    - $(A \cap B)' = A' \cup B'$
- $A \cup B = B \cup A$, $A \cap B = B \cap A$

Counting

Back to the coin flip example:

- If you flip a coin twice, how many outcomes are there in the sample space? 4
- What if you flip a coin three times? 8
- Four times? 16
- What if you flip it 5,000 times?
- We use counting techniques to help us find the number of outcomes for experiments.

Multiplication
If an experiment contains multiple "steps", then the total number of outcomes for the experiment is equal to the product of the number of outcomes in each step.

Example: Flip a Coin Three Times

- Step 1: Flip a Coin. Number of outcomes: 2 (H, T)
- Step 2: Flip a coin again. Number of outcomes: 2 (H, T)
- Step 3: Flip a coin again. Number of outcomes: 2 (H, T)

Total number of outcomes: $2 \times 2 \times 2 = 2^3 = 8$

Example: Website Design
Suppose you are designing a website and you are given the option to choose one of four colors, one of three fonts, and one of three image sizes. How many possible designs for your website are there?

- Step 1: Choose a color. Number of outcomes: 4
- Step 2: Choose a font. Number of outcomes: 3
- Step 3: Choose an image size. Number of outcomes: 3

Total number of outcomes: $4 \times 3 \times 3 = 36$.

Permutations
If you have three elements: a, b, c which can be arranged in any order (abc, acb, bac, etc.), how many possible sequences are there?

- Possible sequences (or permutations): abc, acb, bac, bca, cab, cba (6)
- Another way to find this number is: $3! = 3 \times 2 \times 1 = 6$
- The number of permutations (sequences) of n elements is $n!$

Permutations of Subsets
If you have three elements: a, b, c, and want to find the number of permutations (sequences) of any two elements, how many possible sequences are there?

- Possible permutations: ab, ba, ac, ca, bc, cb (6)
- Another way to find this number is $P_2^3 = 3!/(3-2)! = 6$
- The number of permutations of subsets of r elements selected from a set of n elements is:
$$P_r^n = \frac{n!}{(n-r)!}$$

Example: Printed Circuit Board
Suppose a printed circuit board has 8 locations in which a component can be placed, and you have four different components to place on the board, how many different arrangements are possible?

- We are looking for the permutation of subsets of 4 different components to be placed on 8 different locations
- $P_4^8 = \frac{8!}{(8-4)!} = \frac{8!}{4!} = 8 \times 7 \times 6 \times 5 = 1680$

Permutations of Similar Objects (Example: Hospital Schedule)
Suppose you are scheduling the OR usage for a hospital. You have to schedule 3 knee surgeries (k) with Dr. Knee and 2 hip surgeries (h) with Dr. Hip. How many possible schedule sequences are there for the doctors?

Note: We will treat all knee surgeries as similar to each other since they are all performed by the same doctor. We will do the same for hip surgeries.

- Possible permutations: kkkhh, kkhkh, kkhhk, khkkh, khkhk, khhkk, hkkkh, hkkhk, hkhkk, hhkkk (10)
- Another way to find this number is: $\frac{5!}{3!2!} = 10$
- For a set of n total objects where there are n_1, n_2, …, n_r similar objects, the number of <u>permutations of similar objects</u> is:

$$\frac{n!}{n_1!\,n_2!\dots n_r!}$$

Example: Printed Circuit Board
Let's repeat the printed circuit board example (8 locations for a component) but this time let's suppose there are 5 identical (similar) components to be placed. How many possible designs are there?

- Here we have to think a bit to determine how to solve this problem
- For the circuit board, there will be five slots with a component, and three empty slots
- We will treat the five slots with components as similar and do the same for the empty slots
- In this case, for a set of 8 elements where 5 have a component and 3 are empty, the number of permutations of similar objects is:

$$\frac{8!}{5!\,3!} = 56$$

The Binomial Coefficient and Combinations
- Let's introduce an important operation: the binomial coefficient.
- The number of subsets of r elements that can be selected from a set of n elements is:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

This operation is called the "binomial coefficient" (we'll explain why later) and is usually pronounced "n choose r". The result of this operation is called a <u>combination</u>.

Example: Manufacturing Parts
Suppose you have a set of 3 parts, from which you want to choose 2. How many possible combinations are there?

- We have a set of 3 parts, and we want to choose 2, so "3 choose 2"
- $\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$

Example: Poker Hands
In poker, a standard deck has 52 cards and we choose 5 at random (5 card hand). How many possible combinations of hands can you have?

- We have a deck of 52 cards, and we want to choose 5, so "52 choose 5"

- $\binom{52}{5} = \frac{52!}{5!(52-5)!} = 2{,}598{,}960$

Some Final Definitions

- sampling: selecting an item from a set of items. For example, selecting one part from a batch of parts.
- sampling with replacement: when you select an item from a batch and then place the item back into the batch before selecting a second item.
- sampling without replacement: when you select an item from a batch and then select a second item without replacing the first item

Example: Tools in a Box

If there are 10 tools in a box and you take a tool out of the box, how many tools can you choose from if you select a second tool using:

- sampling with replacement? (10)
- sampling without replacement? (9)

Example: Applying Statistical Thinking to a Manufacturing Problem

Suppose you have a bin with 50 parts of which 3 are defective (and 47 are not defective). You select a sample of 6 parts (without replacement). How many different possible samples are there that contain exactly 2 defective parts?

- Our samples will contain 2 defective parts.
- First let's find out how many possible combinations of the two defective parts are there.
- We want to choose 2 defective parts from the 3 total defective parts:
- $\binom{3}{2} = 3$
- Since each sample has 6 parts, then the other 4 must be non-defective parts.
- We want to choose 4 non-defective parts from the 47 total non-defective parts:
- $\binom{47}{4} = \frac{47!}{4!(47-4)!} = 178{,}365$
- Now, to determine the total number of possible samples with 2 defective parts, we use multiplication
- $3 \times 178{,}365 = 535{,}095$

Sampling Table

	order matters	order doesn't matter
with replacement	n^k	$\binom{n+k-1}{k}$
without replacement	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Homework: Problems 2.1.7, 2.1.13, 2.1.19, 2.1.23, 2.2.3, 2.2.7, 2.2.11

References

Blitzstein, Hwang, *Introduction to Probability*

Montgomery, Runger, *Applied Statistics and Probability for Engineers*