Mathematics for Fluid Mechanics
Module 1

Derivatives

1-1 Terminology and Notation

1-1-1 Functions and Variables

In our study of derivatives, we will refer to dependent variables as functions, which depend on variables. For example:

$$f(x) = 2x \qquad g(x) = x^2 + 4 \quad y(x) = \frac{x^3}{2} + \frac{5}{x}$$

Above, we show three functions, f, g, and y, which depend on the variable x.

$$s(x, t) = 4xt^2 + 2xt$$

Above, we show a function s that depends on both x and t. Functions can depend on any number of independent and dependent variables.

$$v[x(t), t] = x^2 t + x^3$$

$$x(t) = 5t$$

Above, we show a function v that depends on both x and t, but where x is not an independent variable. Instead, x also depends on t, and the second function shows that relationship.

1-1-2 Differentials

The differential, d is an operator that means "a very small change in…" For example, if a body is at position x_1 and then moves to a different position x_2, the change in position can be expressed as:

$$\Delta x = x_2 - x_1$$

In this case Δx means "a change in x." This is a finite change, because it is measurable and quantifiable. However, if we want to look at a very small change in position, that is a change so small that Δx is almost zero, then we can instead use the differential operator:

$$dx$$

In this case, dx means "a very small change in x;" a change so small that we say it is an infinitesimally small change. It is hard to physically explain a differential by itself, but we will learn that we can use this differential operator in conjunction with other differential operators to understand change in functions.

1-1-3 Derivatives

Looking back at our function $y(x)$ above, we can deduce that if we have a very small change in the variable, x, we may then expect to have a very small change in y. The relationship between these changes is called the derivative of y with respect to x:

$$\frac{dy}{dx}$$

The standard way of teaching derivatives in most math courses is through the notion of slopes and limits. You may recall the slope between two points as the "rise over the run." If we graph a curve of our function y with respect to its variable x, then the "rise" is the change in the function, and the "run" is the change in the variable. Therefore, for two points along a curve at x and $x + \Delta x$, the slope is:

$$\frac{\Delta y}{\Delta x} = \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

But now, let's imagine that we want to see the slope of a line tangent to the curve at a point. In this case, we are looking for the slope between two points that are so close to each other that we can't quantify the distance between them. What we are doing, then is looking for a change in rise and run so small that Δx is almost zero. We usually say that Δx "approaches zero." Notice that we can't really set $\Delta x = 0$ because this would leave the slope as an undefined function. Instead, we look at the limit, that is the slope "just before" Δx reaches zero. Mathematically, we write this as:

$$\lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

Note that this limit gives us the slope of a line tangent to the curve at a point x. Also, based on our definition of the derivative, where d represents a very small change, then this limit is also the derivative of the function!

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$

With this definition, we can now determine the derivative for any function!

Example: Let's find the derivative of $y = x^2$ with respect to x.

From our definition of a derivative as a limit, we can write

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

Solving:

$$\frac{d}{dx}(x^2) = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x$$

Now, as Δx approaches zero:

$$\frac{d}{dx}(x^2) = 2x + 0 = 2x$$

So, the derivative of $y = x^2$ with respect to x is $2x$!

1-2 Derivatives of Basic Functions

There are basic functions for which their derivatives are commonly known. These are usually derived in a Calculus course using the limit definition we have introduced. While we will not derive every function covered in this module, there are some derivatives that will be useful to know for any engineering course:

1-2-1 Derivative of a Function times a Constant

Let $g(x) = kf(x)$ where k is a constant and $f(x)$ is a function of x. The derivative of $g(x)$ is:

$$\frac{dg}{dx} = \frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$$

We will prove this in a future lesson, but knowing this will be helpful when deriving functions. Colloquially, we sometimes say that we can "take the constant out of the derivative."

1-2-2 Derivative of a Power Function

Let $f(x) = x^n$ where n is a constant. The derivative of $f(x)$ is:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

1-2-3 Derivative of an Exponential Function

Let $f(x) = a^x$ where a is a constant and $a > 0$. The derivative of $f(x)$ is:

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

From this rule, we can also show that

$$\frac{d}{dx}(e^x) = e^x \ln(e) = e^x$$

1-2-4 Derivative of a Logarithmic Function

Let $f(x) = \ln(x)$ where $x > 0$. The derivative of $f(x)$ is:

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

1-2-5 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\tan(x)] = 1 + \tan^2(x) = \sec^2(x)$$

1-2-6 Derivatives of Inverse Trigonometric Functions

For $-1 \leq x \leq 1$:

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1 - x^2}}$$

For $-1 \leq x \leq 1$:

$$\frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1 + x^2}$$

1-3 Rules of Derivation

These rules of derivation, in conjunction with the derivatives of basic functions shown above can help us determine the derivative for virtually any function or combination of functions.

1-3-1 Derivative of a Constant

Let $f = k$ where k is a constant. The derivative of f is:

$$\frac{d}{dx}(k) = 0$$

1-3-2 Sum Rule

Let $H(x) = f(x) + g(x)$. The derivative of $H(x)$ is:

$$\frac{dH}{dx} = \frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

1-3-3 Product Rule

Let $H(x) = f(x)g(x)$. The derivative of $H(x)$ is:

$$\frac{dH}{dx} = \frac{d}{dx}[f(x)g(x)] = g\frac{df}{dx} + f\frac{dg}{dx}$$

1-3-4 Chain Rule

Let $H(x) = f[g(x)]$. This means that f is a function of g, which itself is a function of x. The derivative of $H(x)$ is:

$$\frac{dH}{dx} = \frac{d}{dx}\{f[g(x)]\} = \frac{df}{dg}\frac{dg}{dx}$$

1-3-5 Quotient Rule

We can use the chain rule and the product rule to obtain the quotient rule. Let $H(x) = f(x)/g(x)$. The derivative of $H(x)$ is:

$$\frac{dH}{dx} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

1-4 Partial Derivatives

1-4-1 Introduction

Let $f = f(x, y)$. In this case, where the function depends on more than one variable, how do we find df/dx? What about df/dy?

We introduce the concept of a partial derivative of a function. That is, the change of that function with respect to only one of its variables. If $f = f(x, y)$ and we want to find the partial derivative of f with respect to x, then we treat x as the only variable in the function. That is, we treat all other variables as constants. We use a special symbol ∂ for a partial derivative, such as:

$$\frac{\partial f}{\partial x}$$

Example: Let's find the partial derivative of $f(x, y) = x^2y$ with respect to x. Remember that we will treat all other variables (in this case, y) as constants.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y) = 2xy$$

Let's now fund the partial derivative of f with respect to y. In this case, we will treat x as a constant.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y) = x^2$$

1-4-2 Total Change and Partial Change

Now that we have introduced partial derivatives, let's revisit our definition of a differential (a very small change) for the case of a multivariable function. Let $f = f(x, y)$. What does a small change in f mean in terms of x and y?

A very small change in f can be the result of a very small change in x and a very small change in y. In this case, the rate at which f changes with respect to x is not the same as the rate at which it changes with respect to y. Therefore, to consider both variables, we can define a small change in f as:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

This means that the small change in f will be defined by the small change in x as well as the small change in y. Note that here we are using the differential operator d to represent a small change, and the partial operator ∂ to represent a partial derivative. To distinguish between the two, the differential (not partial) change is usually called the "total change."

This can apply to as many variables as the function has. For example, if $f = f(x, y, z)$, then a small change in f will be:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

1-4-3 Chain Rule for Multivariable Functions

We now revisit the chain rule. For the total change introduced above, if all variables are independent then the derivative of f with respect to x will be:

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}$$

Notice that $dx/dx = 1$, and since the variables are independent of each other, then $dy/dx = dz/dx = 0$. However, what if the variables are not independent?

According to the chain rule, if $f = f[x, y(x)]$, then the derivative of f with respect to x is:

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial y}{\partial x}\frac{dy}{dx}$$

Notice that this shows a distinction between df/dx and $\partial f/\partial x$. To distinguish the two, we call the former the total or ordinary derivative.

1-5 Application: Acceleration and the Material Derivative

Let's apply what we have learned to study a fundamental concept in fluid mechanics: the material derivative.

Consider a fluid particle moving (flowing) along a curve. In cartesian coordinates, its velocity can be seen as a function of position (x, y, z) and time (t). However, the position of the fluid particle is also a function of time. Furthermore, the velocity of a fluid particle, having both magnitude and direction, is a vector. Thus, we write:

$$\vec{\mathrm{v}}[x(t), y(t), z(t), t] = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

Where we call the velocity components in the x, y, and z directions, u, v, and w, respectively. Note also that the velocity components themselves can be related to the position and time of the particle by:

$$u = \frac{dx}{dt}$$

$$v = \frac{dy}{dt}$$

$$w = \frac{dz}{dt}$$

Now, given all this information, what is the acceleration of a fluid particle? In other words, what is:

$$\vec{a} = \frac{d\vec{\mathrm{v}}}{dt}$$

Note that we are looking for a total change in velocity with respect to time, not a partial change.

Since this is a multivariable function, we apply the chain rule to obtain a small change in velocity $d\vec{\mathrm{v}}$:

$$d\vec{\mathrm{v}} = \frac{\partial\vec{\mathrm{v}}}{\partial x}dx + \frac{\partial\vec{\mathrm{v}}}{\partial y}dy + \frac{\partial\vec{\mathrm{v}}}{\partial \mathrm{z}}dz + \frac{\partial\vec{\mathrm{v}}}{\partial t}dt$$

Then, the derivative of velocity with respect to time is:

$$\frac{d\vec{\mathrm{v}}}{dt} = \frac{\partial\vec{\mathrm{v}}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{\mathrm{v}}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{\mathrm{v}}}{\partial \mathrm{z}}\frac{dz}{dt} + \frac{\partial\vec{\mathrm{v}}}{\partial t}\frac{dt}{dt}$$

Note that $dx/dt = u$, $dy/dt = v$, $dz/dt = w$, and $dt/dt = 1$, so we can write that.

$$\frac{d\vec{\mathrm{v}}}{dt} = u\frac{\partial\vec{\mathrm{v}}}{\partial x} + v\frac{\partial\vec{\mathrm{v}}}{\partial y} + w\frac{\partial\vec{\mathrm{v}}}{\partial \mathrm{z}} + \frac{\partial\vec{\mathrm{v}}}{\partial t}$$

This is the acceleration of a fluid particle. This type of derivative (the change in a property through space and time) is a very important fluid mechanics derivative known as the material derivative, and it goes by a special notation D/Dt, so that:

$$\vec{a} = \frac{D\vec{\mathrm{v}}}{Dt} = u\frac{\partial\vec{\mathrm{v}}}{\partial x} + v\frac{\partial\vec{\mathrm{v}}}{\partial y} + w\frac{\partial\vec{\mathrm{v}}}{\partial \mathrm{z}} + \frac{\partial\vec{\mathrm{v}}}{\partial t}$$

Here, we say that acceleration is the material derivative of velocity. We can apply the material derivative to any other property that varies with space and time. For example, the material derivative of a fluid's density is:

$$\frac{D\rho}{Dt} = u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial \mathrm{z}} + \frac{\partial\rho}{\partial t}$$